Wave scattering through classically chaotic cavities in the presence of absorption: An information-theoretic model

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We propose an information-theoretic model for the transport of waves through a chaotic cavity in the presence of absorption. The entropy of the S-matrix statistical distribution is maximized, with the constraint $\langle \mathrm{Tr}\, SS^{\dagger} \rangle = \alpha n$: n is the dimensionality of S, and $0 \leq \alpha \leq 1$, $\alpha = 0(1)$ meaning complete (no) absorption. For strong absorption our result agrees with a number of analytical calculations already given in the literature. In that limit, the distribution of the individual (angular) transmission and reflection coefficients becomes exponential (Rayleigh statistics), even for $n = 1$. For $n \gg 1$ Rayleigh statistics is attained even with no absorption; here, we extend the study to $\alpha < 1$. The model is compared with random-matrix-theory numerical simulations: it describes the problem very well for strong absorption, but fails for moderate and weak absorptions. Thus, in the latter regime, some important physical constraint is missing in the construction of the model.

PACS number(s): 05.45.−a, 42.25.Bs, 41.20.Jb

Systems involving multiple elastic scattering of any kind of waves (sound, microwaves, or light), whose interference leads to strong fluctuations in the transmitted intensity, can be described in very much the same way as electron systems. However, there exists an important difference: the interference pattern for classical waves can be affected as a result of loss or *absorption*, which is absent in electron systems. The issue is extremely important from an experimental point of view, because absorption is always present and is often very strong. For diffusive transport the problem was intensively studied both experimentally [1] and theoretically [2–4]. The issue has also attracted attention in connection with the phase-coherent reflection of light by a disordered medium which amplifies radiation [5,6] and the study of the relation between absorption and dephasing [4,7]. The analytical evaluation of the reflection-matrix statistical distribution for a semi-infinite disordered waveguide was performed, for arbitrary absorption, in Ref. [5], and that of the S-matrix distribution for a chaotic cavity with absorption and one propagating mode in each of two waveguides, in Ref. [7].

In the present paper we take up again the problem of the propagation of scalar waves traveling through a cavity, whose classical dynamics would be chaotic, in the presence of absorption, connected to the outside through a number of waveguides supporting an *arbitrary* number of propagating modes. Motivated by the success of an information-theoretic approach [8] to the study of chaotic scattering through cavities, we propose below an extension of such models to study the effect of absorption. Since these models are based on the idea of doing statistics directly on the S matrix of the system (on the basis of the information which is physically relevant for the problem in question) we believe that the present approach complements the analytical derivations mentioned above. We show that the two approaches agree in the limit of strong absorption, while for moderate and weak absorption

some relevant information is missed in our model.

The scattering of waves through a cavity can be described by an S matrix that relates incoming and outgoing amplitudes. The dimensionality n of the matrix is the total number of channels in all the waveguides. For two N-channel waveguides, $n = 2N$ and the S matrix has the structure

$$S = \begin{bmatrix} r & t' \\ t & r' \end{bmatrix}, \tag{1}$$

where r, r' and t, t' are the N-dimensional matrices of reflection and transmission amplitudes with incidence from either waveguide.

In quantum mechanics, the universality classes for S matrices were introduced by Dyson [9,10]. In the absence of any symmetries, the only restriction on S is unitarity, $SS^{\dagger} = I$, due to flux conservation (the *unitary* or $\beta = 2$ case). In the *orthogonal* case ($\beta = 1$), S is symmetric because of either time-reversal invariance (TRI) and integral spin, or TRI, half-integral spin, and rotational symmetry. In the *symplectic* case ($\beta = 4$), S is self-dual because of TRI with half-integral spin and no rotational symmetry. The intuitive idea of *equal-a-priori probabilities* is expressed mathematically by the *invariant measure* on the matrix space under the symmetry operation for the class in question, giving the *circular* orthogonal, unitary, and symplectic ensembles (COE, CUE, CSE).

Consider the orthogonal case. The potential appearing in the Schrödinger equation is real, with a strength u_0, say. The resulting S matrix is unitary, $SS^{\dagger} = I$, and symmetric, $S = S^T$. Suppose we analytically continue u_0 to complex values: $u_0 = u_0' - iu_0''$, the sign of the imaginary part ensuring absorption. Because of loss of flux, the resulting S is now *subunitary*, meaning that the eigenvalues of the Hermitian

matrix $h = SS^\dagger$ lie in the interval between zero and one. However, the *symmetry* property, $S = S^T$, is not altered. We still speak of the orthogonal ($\beta = 1$) case. For the unitary one, a similar analytic continuation gives a subunitary and in general nonsymmetric S matrix. In the scattering problem of scalar classical waves, the orthogonal case is the physically relevant one. However, we deal below with both $\beta = 1$ and 2, the unitary case being presented as a reference problem, as it is often simpler to treat mathematically than the orthogonal one.

Following Ref. [11], we introduce a *uniform weight in the space of sub-unitary matrices* as

$$d\mu_{sub}^{(\beta)}(S) = C\theta(I - SS^\dagger)\prod_{a,b} dx_{ab}dy_{ab}, \quad (2)$$

where $S_{ab} = x_{ab} + iy_{ab}$; $\Pi_{a,b}$ is over all elements in the unitary case, but, in the orthogonal one, only over $a \le b$. In this equation, the step function $\theta(H)$ (for a Hermitian H) is nonzero for $H > 0$ (i.e., for H positive definite, so that all its eigenvalues are positive definite) and thus selects *subunitary* matrices.

A complex $n \times n$ subunitary matrix can be written in the *polar representation* (Refs. [11] and [7]) as

$$S = UDV. \quad (3)$$

The unitary matrices U and V are arbitrary in the unitary case, while $V = U^T$ in the orthogonal one. The matrix D is diagonal, with the structure $D_{ab} = \rho_a \delta_{ab}$ ($a = 1, \ldots, n$), with $0 \le \rho_a \le 1$.

The explicit expression of the above measure (2) in terms of the independent parameters of the polar representation (3) is (Ref. [11])

$$d\mu_{sub}^{(\beta)}(S) = \prod_{a<b}|\rho_a^2 - \rho_b^2|^\beta \prod_c \rho_c d\rho_c d\mu(U)d\mu(V), \quad (4)$$

where $d\mu(U)$ and $d\mu(V)$ are the invariant measure for the unitary group in n dimensions; $d\mu(V)$ is absent in the orthogonal case $\beta = 1$.

More general statistical distributions of subunitary matrices carrying more information than the equal-*a-priori* probability (2) can now be constructed using $d\mu_{sub}(S)$ as a starting point and writing

$$dP(S) = p(S)d\mu_{sub}(S). \quad (5)$$

In what follows we propose an information-theoretic criterion to choose $p(S)$. The *information-theoretic entropy* $\mathcal{S}$ of the S-matrix distribution [12], $\mathcal{S}[p(S)] = -\int p(S)\ln p(S)d\mu_{sub}(S)$, is *maximized subject to the constraint of a given average strength of the absorption*. Mathematically, this is expressed by the average departure from unitarity of our S matrices; we thus write the constraint as

$$\langle \text{Tr } SS^\dagger \rangle = \alpha n, \quad 0 \le \alpha \le 1. \quad (6)$$

Thus, $\alpha = 0$ corresponds to complete absorption and $\alpha = 1$ to lack of absorption. We find

$$dP^{(\beta)}(S) = Ce^{-\nu \text{ Tr } SS^\dagger}d\mu_{sub}^{(\beta)}(S), \quad (7)$$

where the constant C and the Lagrange multiplier ν ensure normalization and the fulfillment of the constraint (6). The limit of no absorption, $\alpha = 1$, is attained when the Lagrange multiplier $\nu \to -\infty$ and the distribution concentrates on the unitarity sphere. The limit of complete absorption, $\alpha \to 0$, is attained when $\nu \to +\infty$: the distribution then becomes a δ function at the origin and there is no exit signal. The result of Eq. (7), a Laguerre ensemble for the variables ρ_a^2, coincides, for strong absorption, with that obtained in Ref. [5] for the diffusive waveguide and in Ref. [7] for cavities with $N = 1$.

The "ansatz" (7) entails a number of properties and restrictions. From Eqs. (4) and (3), and the properties of $d\mu(U)$ [13], we see that, under the distribution (7), the average $\langle S \rangle = 0$. Therefore, applications of the model (7) should be restricted to cases where *prompt processes* are absent and so $\langle S \rangle = 0$ [12,14,8]. We shall see at the end of the paper that for uniform volume absorption (in quantum mechanics, a constant potential $-iW$ throughout the cavity) the S matrix can be obtained from that without absorption, evaluating it at the complex energy $E + iW$, i.e., $S(E + iW)$. The so-called analyticity-ergodicity requirements $\langle S_{a_1b_1}^{n_1}\cdots S_{a_pb_p}^{n_p}\rangle = \langle S_{a_1b_1}\rangle^{n_1}\cdots\langle S_{a_pb_p}\rangle^{n_p}$ are discussed in Ref. [12] for unitary matrices: the same argument applies here as well, and the necessity to fulfill them follows. In the present case, their fulfillment follows from Eqs. (7), (4), and (3), and the properties of $d\mu(U)$ [13].

As a brief excursion into the situation when $\langle S \rangle \ne 0$, we note that, as W increases, $S(E + iW) \to \langle S \rangle$, which, in turn, is the so called "optical S matrix," that signals the presence of prompt or direct processes [12,14,8]. As a consequence, S, for large absorption, does not tend to zero, but to $\langle S \rangle$. With no absorption, the splitting $S = \langle S \rangle + S^{fl}$ describes the problem in terms of two responses, associated with two distinct time scales: the prompt and the equilibrated one. It is then expected that strong absorption will affect the former response, that corresponds to short trajectories, much less then the latter one, that arises from long chaotic trajectories.

We analyze some of the consequences of the ansatz (7) and, at the end, compare them with the results of random-matrix-theory (RMT) numerical simulations.

a. The $n = 1$ case. This case, which describes a cavity with one waveguide supporting only one open channel (S is thus the reflection amplitude back to the only channel we have), is, within our model (7), independent of the universality class β. Equation (3) for S in the polar representation reduces to $S = \rho \exp i\theta$; ρ^2 represents the reflection coefficient R. The uniform weight (4) and the distribution (7) reduce to

$$d\mu_{sub}(S) = \rho d\rho d\theta, \quad dP(S) = Ce^{-\nu\rho^2}\rho d\rho d\theta. \quad (8)$$

The R-probability density is

$$w(R) = De^{-\nu R}, \quad 0 \le R \le 1, \quad (9)$$

D and ν being given by

$$D = \frac{\nu}{1 - e^{-\nu}}, \quad \langle R \rangle = \frac{1}{\nu} - \frac{1}{e^\nu - 1} = \alpha. \quad (10)$$
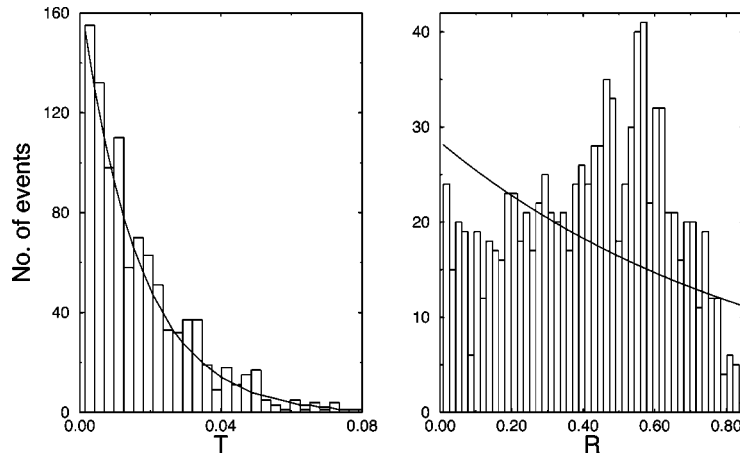
FIG. 1. Distribution of the reflection coefficient R for a cavity with one waveguide, supporting one channel: (left) in the presence of strong absorption ($\langle R \rangle = \alpha = 0.034$); (right) for moderate absorption ($\langle R \rangle = \alpha = 0.410$).

For weak absorption, $\alpha \approx 1$, $\nu \to -\infty$ and the distribution (9) becomes strongly peaked around $R = 1$, i.e., the unitarity circle, reducing to the one-sided delta function $\delta(1 - R)$ as $\alpha \to 1$. In the other extreme of *strong absorption*, $\nu \to +\infty$, $\alpha \approx 1/\nu$, $D \approx 1/\alpha$ and

$$w(R) \approx \langle R \rangle^{-1} e^{-R/\langle R \rangle}, \tag{11}$$

Rayleigh's distribution, with the average $\langle R \rangle = \alpha$.

b. The orthogonal case for arbitrary n. Using the results of Ref. [13] we find, for the average of an individual (angular) transmission or reflection coefficient

$$\langle T_{ab} \rangle^{(1)} = \langle R_{a \neq b} \rangle^{(1)} = \alpha/(n+1) = (1/2)\langle R_{aa} \rangle^{(1)}. \tag{12}$$

We see the occurrence of the familiar *backward enhancement factor 2*. For the second moments we find

$$\langle T_{ab}^2 \rangle^{(1)} = \langle R_{a \neq b}^2 \rangle^{(1)} = \sum_{\alpha=1}^{n} \langle |U_{1\alpha}|^4 |U_{2\alpha}|^4 \rangle_0 \langle \rho_\alpha^4 \rangle^{(1)}$$

$$+ 2 \sum_{\alpha \neq \gamma = 1}^{n} \langle |U_{1\alpha}|^2 |U_{2\alpha}|^2 |U_{1\gamma}|^2 |U_{2\gamma}|^2 \rangle_0 \langle \rho_\alpha^2 \rho_\gamma^2 \rangle^{(1)}. \tag{13}$$

The indices 1 and 2 indicate any pair of different channels; $\langle \cdots \rangle_0$ stands for an average with respect to the invariant measure of the unitary group [13]. For $\langle R_{aa}^2 \rangle^{(1)}$ one sets the two indices 1 and 2 equal.

For a two-waveguide problem with one channel in each waveguide the S matrix is two dimensional ($n = 2N = 2$). Equation (12) gives $\langle T \rangle^{(1)} = \alpha/3, \langle R \rangle^{(1)} = 2\alpha/3$. Restricting ourselves to the limit of *strong absorption*, $\alpha \ll 1$, the Lagrange multiplier $\nu = 3/2\alpha$ and we obtain

$$\langle T^2 \rangle^{(1)} \approx 2[\langle T \rangle^{(1)}]^2. \tag{14}$$

Although we have not calculated the statistical distribution of T, result (14) is consistent with the Rayleigh distribution for the transmission coefficient T.

For a large number of channels, $n = 2N \gg 1$, Eqs. (13) and (12) give [13]

$$\langle T_{ab}^2 \rangle^{(1)} = \langle R_{a \neq b}^2 \rangle^{(1)} \approx 2 \frac{\alpha^2}{n^2} \approx 2[\langle T_{ab} \rangle^{(1)}]^2, \tag{15}$$

and similarly, $\langle R_{aa}^2 \rangle^{(1)} \approx 2[\langle R_{aa} \rangle^{(1)}]^2$, the relation between first and second moments for an exponential distribution with the average value (12), which becomes smaller as the absorption increases.

For no absorption one reaches, in the limit $n \gg 1$, a Rayleigh distribution for R_{aa}. Reference [15] shows that, for the COE ($\approx$ being an approximation for large n)

$$w^{(1)}(R_{aa}) = C(1 - R_{aa})^{(n-3)/2} \approx \langle R_{aa} \rangle^{-1} e^{-R_{aa}/\langle R_{aa} \rangle}.$$

c. The unitary case for arbitrary n. In the unitary case, the statistical properties of a transmission coefficient is identical to that of a diagonal or off-diagonal reflection coefficient. The following equations are thus written for T_{ab}. We find, for its average, $\langle T_{ab} \rangle^{(2)} = \alpha/n$; we have used the result [13] $\langle |U_{1\alpha}|^2 \rangle_0 = 1/n$. The difference in expectation values between the two symmetry classes is thus

$$\langle T_{ab} \rangle^{(1)} - \langle T_{ab} \rangle^{(2)} = -\alpha/[n(n+1)]. \tag{16}$$
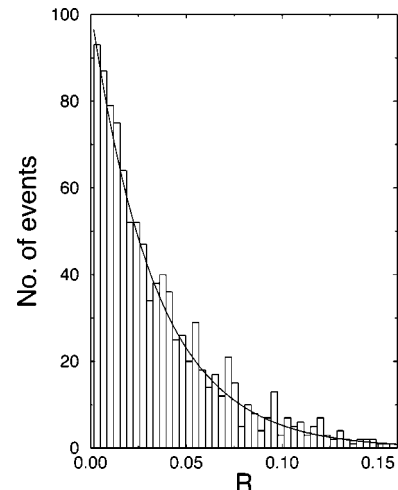

FIG. 2. Distribution of the transmission coefficient T for a cavity with two waveguides, each supporting one open channel, in the presence of strong absorption ($\alpha = .049$).

For the second moment of T_{ab} we have

$$\langle T_{ab}^2 \rangle^{(2)} = 2 \sum_{\alpha \gamma} \langle |U_{1\alpha}|^2 |U_{1\gamma}|^2 \rangle_0 \langle |V_{\alpha 2}|^2 |V_{\gamma 2}|^2 \rangle_0 \langle \rho_\alpha^2 \rho_\gamma^2 \rangle$$

$$= [2/(n+1)^2]\{[(n-1)/n]\langle \rho_1^2 \rho_2^2 \rangle^{(2)} + (2/n)\langle \rho_1^4 \rangle^{(2)}\}. \quad (17)$$

We have used the result [13] $\langle |U_{1\alpha}|^2 |U_{1\gamma}|^2 \rangle_0 = (1 + \delta_{\alpha\gamma})/[n(n+1)]$.

For a two-waveguide problem with one channel in each waveguide ($n=2N=2$) we have $\langle T \rangle^{(2)} = \alpha/2$. In the limit of *strong absorption*, $\alpha \ll 1$, the Lagrange multiplier $\nu = 2/\alpha$ and we obtain a relation like (14). For a large number of channels $n=2N \gg 1$, Eq. (17) gives a similar relation, now for T_{ab}, which is again consistent with Rayleigh's distribution.

For CUE (i.e., no absorption) one reaches a Rayleigh distribution for $n \gg 1$. Pereyra and Mello [15] find the distribution of a single transmission coefficient to be

$$w(T_{ab}) = C(1-T_{ab})^{n-2} \approx \langle T_{ab} \rangle^{-1} e^{-T_{ab}/\langle T_{ab} \rangle}. \quad (18)$$

d. Comparison with numerical simulations. Some of our predictions are compared below with RMT numerical simulations for $\beta = 1$. The S matrices are constructed as $S(E) = -[I_n - iK(E)]^{-1}[I_n + iK(E)]$, with $K_{ab}(E) = \Sigma_\lambda \gamma_{\lambda a} \gamma_{\lambda b}/[E_\lambda - E]$ and $(I_n)_{ab} = \delta_{ab}$ ($a,b = 1, \ldots, n$). The E_λ's are generated from an ''unfolded'' zero-centered GOE [16] with average spacing Δ. The $\gamma_{\lambda a}$'s are statistically independent, real, zero-centered Gaussian random variables. At $E=0$, $\langle S_{ab} \rangle = -[1 + \pi \langle \gamma_{\lambda a}^2 \rangle / \Delta]^{-1}[1 - \pi \langle \gamma_{\lambda a}^2 \rangle / \Delta]\delta_{ab}$, and we require $\langle S \rangle = 0$. In the quantum case, addition of a constant imaginary potential $-iW$ inside the cavity makes the E_λ's complex and equal to $E_\lambda - iW$ (see also Ref. [7]). This is equivalent to evaluating the above expressions at the complex energy $E+iW$, which makes $S(E+iW)$ subunitary. Although Eq. (7) gives the probability distribution for

the full S matrix and arbitrary n, we only analyze below individual (angular) reflection and transmission coefficients, for $n=1,2$.

Figure 1 shows for $n=1$ the results of the RMT numerical simulations (as histograms), compared with the present model for the corresponding value of α (continuous curves). For strong absorption the model works very well, the agreement with Rayleigh's law being excellent, while for moderate and weak absorptions the model fails.

Figure 2 shows the distribution of the transmission coefficient T obtained from a RMT simulation for $n=2$. The agreement with the Rayleigh distribution with the centroid $\langle T \rangle = \alpha/3$ is excellent; $\langle R \rangle$ was also checked and found to agree with $2\alpha/3$.

That individual transmission and reflection coefficients attain a Rayleigh distribution for strong absorption can be understood as follows. $S_{ab}(E+iW)$ coincides [8,12] with the energy average of $S_{ab}(E)$ evaluated with a Lorentzian weighting function of half-width W. If Γ^{corr} is the correlation energy, W can be thought of as containing $\sim m = W/\Gamma^{corr}$ independent intervals. If $m \gg 1$, by the central-limit theorem the real and imaginary parts of S_{ab} attain a Gaussian distribution, and $|S_{ab}|^2$ an exponential distribution. This seems to be the situation captured by the maximum-entropy approach.

In summary, the results presented in this paper indicate that wave scattering through classically chaotic cavities in the presence of strong absorption can be described in terms of an information-theoretic model.

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